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***More than you ever wanted to know about Significant Digits and Rounding,
for Students in Statistics and Econometrics Classes***

Most numbers used in these classes are real or hypothetical measurements of social and economic variables, or numbers calculated from such measurements. As such, *they are only as precise as the measurement process that generated them.*

If a math test is scored to the nearest whole number, it does not really make sense to say, on the basis of some regression analysis, that one year of education increases the expected score by 12.753439209…. Yet many students have a tendency to give answers in problem sets and on tests with as many places after the decimal as is shown in their calculators. I have seen pages and pages covered with extremely long decimals. This is a waste of time, both yours and your professor's. It also increases the risk of error.

More importantly, reporting too many digits implies a greater precision in the answer than is justified by the underlying measurements. In that sense, the practice of reporting meaningless digits is actually incorrect and misleading. When the calculator gives a 12-digit answer, only some of those digits are meaningful. Digits that actually convey information are known as "significant digits." Therefore, you should do the following:

*   Learn to recognize how many significant digits you have in the input data.
*   Learn how to determine the number of significant digits you have after you perform your calculations.
*   Round off final answers as needed.

> Quick and dirty bottom line: rarely, if ever, should a final homework answer have more than 3 or 4 significant digits (defined below).

A. How many significant digits do you have to start with?

A.1   In the numbers you are given to work with (not results of your calculations), *non-zero digits (1-9) are always significant*. 123, 12.3 and 1.23 all have three significant digits. (Exception: output from regressions; see A.6.)

A.2   *Zeros before other digits are not significant*, therefore 123, 0.123, 0.0000123 all have three significant digits.

A.3   *Zeros between non-zero digits are significant*. 1002, 3.004 and 0.05006 all have four significant digits.

A.4a   *Zeros at the end of a number, but after the decimal, are there for a reason: they are significant*. 0.2, 0.20, 0.200, 0.2000 all represent the same number, so there is no reason to give the zeros at the end *except to imply that the number is measured*

with a certain level of precision. Explanation: My scale at home gives digital readout rounded to whole pounds. If it says 218, my weight could actually be anywhere between 217.5 and 218.5 (a range of a full pound). If I had a more precise scale, it might give my weight as 218.0000, which would mean that my weight was somewhere between 217.99995 and 218.00005 (a much smaller range!). So 218.0000 is much more precise than 218. They are the same number, but they imply different things about the range of uncertainty that surrounds the number.

A.4b   Zeros *at the end of a number, but before the decimal* (whether the decimal point is shown or not) *are ambiguous*, and must be interpreted by the context. So, 123,000,000 could have as few as 3 and as many as 9 significant digits. Explanation: There is no way to tell just by looking. It depends on the real degree of precision. Has this number been rounded off to the nearest million (anything between 122,500,001 and 123,499,999)? Or is it an exact count? (Note: scientific notation can be used to clarify such numbers, e.g. 1.230×10^8 implies exactly four significant digits. See <u>Scientific Notation</u>, below.)

A.5   *Integers that are exact counts of things are considered to have infinite precision* (therefore an infinite number of significant digits). If a woman has one child, we can write that as 1, but we mean 1.00000000000… exactly, not some number between 0.50001 and 1.49999 that has been rounded off to one. However, an integer number like 1, 2, or 3 can also be approximate, basically a continuous number rounded to zero decimal places. How much does a cup of coffee cost at Starbucks? About $2 (one significant digit). You need to use your judgment and the context of the problem to decide whether an integer is an exact count or an approximate quantity.

A.6   Coefficients reported by regressions are tricky (since they rely on complex calculations on many variables with varying precisions), but assume 3 or at most 4 significant digits. Do not be guided by how many digits a regression program prints out. (After a regression, the statistical uncertainty due to random sampling probably is more important than measurement precision.)

<u>B. How many significant digits do you end up with after calculations?</u> The following rules work well most of the time. They are a substitute for a deeper analysis that calculates an upper and lower bound for the answer based on an upper and lower bound for each number.

B.1   *When multiplying and dividing, including roots and powers, the number of significant digits in the answer is the equal to the least number of significant digits in the inputs.* (This rule is a topic of some debate among mathematicians, with some arguing that it loses information in some cases, but this rule is in general use and is certainly good enough for our purposes.) Round off (see <u>Rounding Rules</u> below) to that number of digits in the final answer.

Examples:

       45.6 * 0.1234 = 5.63 (5.62704 rounded to 3 significant digits)

       0.456/0.0078 = 58 (58.461… rounded to 2 significant digits)

       0.789/3 = 0.3 if 3 is 1 significant digit, but 0.263 if 3 is an exact count (integer)

       sqrt(7.0) = 2.6 (the zero is significant, otherwise it would have been written as 7)

       sqrt(1000) = ? It depends. 31.62 if 1000 has 4 significant digits, etc.

B.2   Adding and subtracting are a little more complicated. Consider each power of ten (ones, tens, hundreds, thousands, etc., on the left of the decimal and tenths, hundredths, thousandths, etc., on the right of the decimal.) This might be easier to visualize if the numbers are written in a decimal-aligned vertical column. *No power of ten is significant in the answer if it includes a digit after (to the right of) the last significant digit in any of the figures being added or subtracted.*

Example:

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   12.36
    4.6
+   0.789
   17.749
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$\rightarrow$ 17.**7** (after rounding to the tenths place)

From this you can see that the tenths place is the rightmost column that is significant. The last significant digit is highlighted in each number. Explanation: The reason is that there is total uncertainty about the value of the digit in the column (power of ten) just to the right of the last significant figure. For example 4.**6** could be 4.55, 4.56,…, 4.64, or 4.65 (see <u>Rounding Rules</u>, below). The uncertainty in that decimal place propagates to the answer when adding or subtracting, and so digits in that place and beyond cannot be significant in the answer. If the hundredths place is uncertain, so is the thousandths and anything smaller. Note that we have three significant digits in the answer even though one of the numbers only has two significant digits. Addition is different than multiplication and division in this regard.

More examples:

5.**1** + 0.456789 = 5.**6**, not 5.556789. (Answer has only 1 place after the decimal.)

12**3**,000,000 + 0.78**9** = 12**3**,000,000 (Millions is the last significant power of ten.)

1 + 0.057**9** = 1.057**9** if 1 is an integer (1.0000…), else just **1**.

In a subtraction, the number of significant figures may plummet: 1.2222 – 1.2221 = 0.0001, with only one significant digit! This problem can wreak havoc with some numerical calculations!

B.3a   When taking a logarithm, the answer should contain as many significant figures after the decimal point as in the argument (the original number). Examples: $\ln(9.87) = 2.289$, $\log_{10}(3.00)=0.477$.

B.3b   When taking an antilog, the answer should contain as many significant digits as there are after the decimal in the argument. Examples: $e^{2.289} = 9.87$, $10^{0.477} = 3.00$.

B.4   If you are using paper and pencil, carry at least 1 extra (non-significant) digit in any intermediate calculations, but keep track of the number of significant digits for use in subsequent calculations (see Exercise 11 below). *If you are using a calculator or spreadsheet, use the full precision of the internally stored numbers until you get to the end of the calculations.* (Note: even if Excel or a calculator shows a number rounded to a limited number of places, it usually stores the number internally with a much higher level of precision.)

B.5   If you are unsure, report 3 or 4 significant digits when working on homework and test problems and don't worry too much about it.

C. Rounding Rules: Common and Correct. **Common practice is to round 5-9 "up" and 0-4 "down."** *You can stop reading here* if you want, but the common method causes a slight upward bias after many roundings. This is something we should worry about if we were building bridges or nuclear power plants. For those interested, to correct this bias, the more precise and correct procedure is as follows:

		Rounded X	
	X	**Common**	**To Even**
	1.0	1	1
	1.1	1	1
	1.2	1	1
	1.3	1	1
	1.4	1	1
	1.5	**2**	**2**
	1.6	2	2
	1.7	2	2
	1.8	2	2
	1.9	2	2
	2.0	2	2
	2.1	2	2
	2.2	2	2
	2.3	2	2
	2.4	2	2
	2.5	**3**	**2**
	2.6	3	3
	2.7	3	3
	2.8	3	3
	2.9	3	3
Mean	**1.95**	**2.00**	**1.95**

C.1   Round *up* if the digits to the right of the last significant digit begin with a 5 followed by at least one non-zero digit. If X is the significant part, then round X*5000…1* or greater up. In other words, increase the last significant digit in X by 1. (Ignore any leading negative sign, i.e. "increase" means the last digit becomes larger in absolute magnitude.)

C.2   Round down to X if the number is X*4999…* or less. In other words, the answer is just X (no change to the last significant digit).

C.3   If the number is X*5000…* exactly (unlikely), or if the intermediate calculation has only one extra, non-significant digit which is a *5*, then technically you are supposed to *round up if X is odd, and round down if X is even, i.e. always round a 5 to an even number.* This is the only difference between the technically correct rule and common practice. To simplify life, feel free to use the common method and always round up with X*5* or greater, but don't tell your chemistry teacher I said this.

Aside: Why the crazy rule of always rounding to an even number? Examine the table of X and Rounded X below. The method of always rounding a final digit of 5 up creates an upward bias, as shown by the increase in the mean after rounding. If we round 5s up half the time, and down half the time, the mean of rounded X is not biased. By convention, we round up when the next digit is odd and down otherwise.

Examples: Round to three significant digits (non-significant digits in italics):

1.23*4* → 1.23
1.23*49* → 1.23
1.23*51* → 1.24
1.23*500001* → 1.24
1.23*5* → 1.24 (Last significant digit is odd.)
1.22*5* → 1.22 (Last significant digit is even, but 1.23 OK by common practice.)
4.99*51* → 5.00 (Note that you need to keep the two zeros – they are significant digits)
-1.23*9* → -1.24 (So rounding "up" really means away from zero. Look at the digits, ignore the sign.)

D. Scientific Notation. Scientific notation is used to express numbers that would be awkward to express in normal decimal form. The basic form is $\textbf{\textit{a.bc}\ldots \textbf{x } \textbf{10}^{\textit{n}}}$, where $\textit{a}$ is the leftmost significant digit and $\textbf{\textit{bc}\ldots}$ are any number of additional significant digits and $\textit{n}$ is the power of $\textbf{10}$ needed to properly express the number. The $\textbf{x}$ simply is the multiplication operator ("times").

Examples:

$1.23 = \textbf{1.23 x } \textbf{10}^{\textbf{0}}$ (Note: $10^0 = 1$)
$12.3 = 1.23 \text{ x } 10 = \textbf{1.23 x } \textbf{10}^{\textbf{1}}$
$123 = 1.23 \text{ x } 100 = \textbf{1.23 x } \textbf{10}^{\textbf{2}}$
$1230 = 1.23 \text{ x } 1000 = \textbf{1.23 x } \textbf{10}^{\textbf{3}}$
$0.000123 = 1.23 \text{ x } 0.0001 = \textbf{1.23 x } \textbf{10}^{\textbf{-4}}$

Just count the number of places the decimal point has to move left (positive) or right (negative); in the last example, the decimal point has to move four places to the right, hence the power of ten is -4. Numerous guides to manipulating scientific notation are available on the internet. Computer programs sometimes use "e" in place of "x 10". Here "e" stands for "exponent", not the base of natural logarithms. So, in computer speak, $1.23\text{e-}4 = 1.23 \text{ x } 10^{-4} = 0.000123$.

E. Miscellaneous. I think it looks nicer and avoids confusion if a leading zero is included in decimals less than 1. In other words, write "0.123", not ".123". However, this is a matter of taste, not mathematics.

Exercises. Calculate the following and report the answer rounded to the correct number of significant digits.

1. $4.56/32.1 =$

2. $e^{0.1231} =$

3. $1234 - 45.07 =$

4. $\ln(5.99) =$

5. $2/3 =$

6. $0.111/0.6 =$

7. $0.111/0.60 =$

8. $0.117/0.60 =$

9. $0.117/0.600 =$

10. $46.7 + 0.01234 + 123.55$

11. $(6.715-6.70)/4.111 =$

12. $12,345*987 =$

13. $12,345/987 =$

14. $12,345-987 =$

15. $12.345-9.87 =$

Since we are more interested in the logic and procedures of econometrics, and many of the numbers used in class are made up or come from poorly specified models, I am far from obsessive about these issues (despite the evidence to the contrary in the form of this note). But keep these rules in mind and try to get into good habits. Feel free to point out how often I slip up when working examples on the board.

Answers to Exercises.

1. $4.56/32.1 = 0.142$

2. $e^{0.1231} = 1.131$ (Four significant figures after the decimal.)

3. $1234 - 45.07 = 1189$ (unless 1234 is an exact count, then what?)

4. $\ln(5.99) = 1.790$ (Three in the argument → three after the decimal.)

5. $2/3 = 0.667$ (assuming 2 and 3 are exact, and applying Rule B.5 and C.1)

6. $0.111/0.6 = 0.2$

7. $0.111/0.60 = 0.18$ (round to even, but common method = 0.19)

8. $0.117/0.60 = 0.20$ (round to even & common method, and note the trailing zero)

9. $0.117/0.600 = 0.195$

10. $46.7 + 0.01234 + 123.55 = 170.3$ (tenths place is last significant.)

11. $(6.715-6.70)/4.111 = 0.004$ (The subtraction yields the intermediate figure of 0.015, but with only one significant digit (the hundredths place). If that were the final answer, we would round to 0.02 and stop there. But since this is just an intermediate product, we keep the full number for the subsequent division, but then round the final result to one significant digit.)

12. $12,345*987 = 12,200,000$ (Avoid the temptation to report more digits. Exception: if the numbers are exact counts, report all digits.)

13. $12,345/987 = 12.5$

14. $12,345-987 = 11,358$

15. $12.345-9.87 = 2.48$ (Hundredths place is last significant.)