# Cheat Sheet for Differential Calculus

## The Derivative.

A derivative is the somewhat contradictory notion of the rate of change of a function at a specific point. For anything other than a linear function, the slope is always changing, so the derivative is a formula that tells what the slope of *y = f(x)* is as a function of *x*. You can think of it as a formula for the slope of a line that is just tangent to the function. Since there is no change at a specific point, we get to the derivative though the idea of a limit: the limit of the change in *y* over the change in *x* as we make smaller and smaller changes in *x*:

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \left[ \frac{f(x+h) - f(x)}{h} \right] \approx \frac{\Delta y}{\Delta x} \text{ for very small changes in x.}$$

All of the derivatives and properties shown below can be demonstrated by starting with this definition.

## Notation.

There are a wide variety of systems of notation for derivatives, such as:

$$\frac{dy}{dx} = \frac{df(x)}{dx} = \frac{d}{dx} f(x) = f'(x) = D_x y$$

## Basic derivatives.

$\frac{d}{dx} c = 0$ A constant never changes, so the rate of change is zero.

$\frac{d}{dx} x = 1$ A 1 unit increase in x leads to … a 1 unit increase in x!

$\frac{d}{dx} x^2 = 2x$ Like the expansion of the area of a square in two dimensions, each edge being one dimensional.

$\frac{d}{dx}x^3 = 3x^2$ A cube expanding in 3 dimensions, each face is two dimensions.

$\frac{d}{dx}x^n = nx^{n-1}$ Following the pattern, but harder to visualize.

$\frac{d}{dx}cx^n = cnx^{n-1}$ The constant just scales things.

Note application to negative powers: $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^{-1} = -1x^{-2} = \frac{1}{x^2}$.

<u>Rules for Taking Derivatives.</u>

*1. The derivative of a sum is the sum of the derivatives.*

$$\frac{d}{dx}\left[f(x)+g(x)\right] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

The sum can include many terms, but the result is the same.

$$\frac{d}{dx}\sum_{i=1}^{n}\left[f(x_i)\right] = \sum_{i=1}^{n}\left[\frac{d}{dx}f(x_i)\right]$$

Thus, in a polynomial with multiple terms, you take the derivative of each term separately.

$$\frac{d}{dx}\left(a+bx+cx^2\right) = \frac{d}{dx}a + \frac{d}{dx}bx + \frac{d}{dx}cx^2 = b + 2cx$$

2. The derivative of a product, aka *The Product Rule*:

$$\frac{d}{dx}\left[f(x)g(x)\right] = \left[\frac{d}{dx}f(x)\right]g(x) + f(x)\left[\frac{d}{dx}g(x)\right]$$

In other words, the derivative of product of two functions is (the derivative of the first function times the second function) plus (the first function times the derivative of the second function). Example:

$$\frac{d}{dx}\left[\left(7x^2\right)\left(3x+7\right)\right]=\left[\frac{d}{dx}\left(7x^2\right)\right]\left(3x+7\right)+\left(7x^2\right)\left[\frac{d}{dx}\left(3x+7\right)\right]$$
$$=14x\left(3x+7\right))+\left(7x^2\right)\left(3\right)$$
$$=42x^2+98x+21x^2$$
$$=63x^2+98x$$
$$\textit{Confirm...}\left(7x^2\right)\left(3x+7\right)=21x^3+49x^2$$
$$\frac{d}{dx}\left(21x^3+49x^2\right)=63x^2+98x$$

In this case, doing the product first was a little easier, but in many cases it may not be.

3. The *Chain Rule* is incredibly useful for taking the derivative of a function of a function.

$$\frac{d}{dx}f\left(g\left(x\right)\right)=\left(\frac{df\left(x\right)}{dg\left(x\right)}\right)\left(\frac{dg\left(x\right)}{dx}\right)$$

You take the derivative of the "outer" function with respect to the inner function, then the derivative of the inner function with respect to x. For example, $(x-2)^2$ is a function (squaring) of a function (subtracting 2) of x. So:

$$\frac{d}{dx}\left(x-2\right)^2=\left(\frac{d\left(x-2\right)^2}{d\left(x-2\right)}\right)\left(\frac{d\left(x-2\right)}{dx}\right)=\left(2\left(x-2\right)^1\right)\left(1-0\right)=2x-4$$

which you can easily verify by squaring first, then taking the derivate of $x^2-4x+4$.

Another example:

$$\frac{d}{dx}\left(3x^2-2x-7\right)^9=9\left(3x^2-2x-7\right)^8\left(6x-2\right).\text{ Harder to do directly!}$$

4. The *Quotient Rule* follows from the product rule and the chain rule.

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{d}{dx}\left[f(x)g(x)^{-1}\right]$$

$$=\left[\frac{d}{dx}f(x)\right]g(x)^{-1}+f(x)\left[\frac{d}{dx}g(x)^{-1}\right]$$

$$\text{(Just the normal Product Rule.)}$$

$$=\left[\frac{d}{dx}f(x)\right]g(x)^{-1}-f(x)g(x)^{-2}\left[\frac{d}{dx}g[x]\right]$$

$$\text{(Here is where we needed the Chain Rule.)}$$

$$=\frac{\frac{d}{dx}f(x)}{g(x)}-\frac{f(x)\left[\frac{d}{dx}g(x)\right]}{g(x)^2}$$

$$=\frac{\left[\frac{d}{dx}f(x)\right]g(x)-f(x)\left[\frac{d}{dx}g(x)\right]}{g(x)^2}$$

Note the numerator is the same as the product rule except for the minus sign.

<u>Some common functions.</u>

$$\frac{d}{dx}e^x=e^x \quad \text{Very convenient.}$$

$$\frac{d}{dx}e^{f(x)}=e^{f(x)}\frac{d}{dx}f(x) \quad \text{By the Chain Rule.}$$

$$\frac{d}{dx}a^x=a^x\ln a$$

$$\frac{d}{dx}\ln x=\frac{1}{x}=x^{-1}, \quad x>0$$

$$\frac{d}{dx}\log_b x=\frac{1}{x\ln b}$$