Guide to Logarithms and Exponents[1]
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1. Review of the Algebra of Exponents.

Before discussing logarithms, it is important to remind ourselves about the algebra of exponents, also known as powers. Exponents are a compact notation to express multiplication of a number or variable by itself:

$$x^1 = x$$
$$x^2 = x \cdot x$$
$$x^3 = x \cdot x \cdot x$$
$$x^9 = x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x$$

Exponents with negative signs (-*n*) are defined to mean the variable is in the denominator, raised to |n|:

$$x^{-1} = \frac{1}{x^1} = \frac{1}{x}$$
$$x^{-2} = \frac{1}{x^2} = \frac{1}{x \cdot x}$$
$$x^{-3} = \frac{1}{x^3} = \frac{1}{x \cdot x \cdot x}$$
$$x^{-9} = \frac{1}{x^9} = \frac{1}{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}$$

Important fact #1: When two powers of the same underlying variable are *multiplied*, the exponents *add*. Examine the following to see why:

$$x^3 x^4 = (x \cdot x \cdot x)(x \cdot x \cdot x \cdot x) = x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x = x^{(3+4)} = x^7$$

In general:

$$x^c x^d = x^{(c+d)}$$

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Important fact #2: When a power of x is divided by a power of x, the exponents must be subtracted (numerator exponent – denominator exponent), because some of the x's "cancel out" and the answer is what is left over:

$$\frac{x^5}{x^4} = \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x \cdot x} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = x^{5-4} = x$$

$$\frac{x^2}{x^4} = \frac{x \cdot x}{x \cdot x \cdot x \cdot x} = \frac{\cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x} \cdot x \cdot x} = x^{2-4} = x^{-2} = \frac{1}{x^2}$$

In general:

$$\frac{x^c}{x^d} = x^{(c-d)}$$

There is an interesting implication of this property. What is x raised to the zero power? Many have an intuition that it should be zero, but in fact **anything to the zero power is 1**. Don't believe it? Look at this:

$$1 = \frac{x^n}{x^n} = x^{(n-n)} = x^0$$

Exponents are just multiplication, and multiplication "pivots" around 1, not zero. When I say it "pivots" around 1, I mean that a number times its inverse is 1, not 0. (For addition the pivot point is zero, so that x plus its additive inverse, -*x*, equals 0.) Thus, $(x)\left(\frac{1}{x}\right) = 1$ can also be written $x^1 x^{-1} = x^{(1-1)} = x^0 = 1$. Again, any number to the zero power is 1.

Important fact #3: When a power of a variable is itself raised to a power, the exponents must be multiplied, as show below:

$$\left(x^3\right)^4 = (x \cdot x \cdot x)(x \cdot x \cdot x)(x \cdot x \cdot x)(x \cdot x \cdot x) = x^{(3)(4)} = x^{12}$$

In general:

$$\left(x^c\right)^d = x^{cd}$$

2. Definition of Logarithms.

With all the preliminaries out of the way we can finally talk about logarithms. Look at the following word equation.

$$base^{exponent} = value$$

In other words, we raise some number called the base to a power and we get a value. For example, $2^4 = ?$ Well, it's not hard to see that the answer is 16. But suppose we knew the answer, but not the exponent: $2^? = 16$. In this case, we are looking for the power to which the number has to be raised to get 16. The answer is 4.

Logarithms are just exponents. All logarithms are answers to questions in this form: to what power must the base be raised to get the given number. Here is how we write the same question in two different ways: $2^? = 16$, or $\log_2(16) = ?$ They both say the same thing: to what power does 2 have to be raised to get 16? So, to repeat, a logarithm is just an exponent. Here are some examples:

$$\log_2(16) = 4 \text{ because } 2^4 = 16$$
$$\log_4(16) = 2 \text{ because } 4^2 = 16$$
$$\log_3(27) = 3 \text{ because } 3^3 = 27$$
$$\log_{10}(10,000) = 4 \text{ because } 10^4 = 10,000$$
$$\log_5(5^2) = 2 \text{ because } 5^2 = 5^2$$

The last one seems a little pointless, but I'll come back to it.

Logarithms are functions, so you really should write them in function notation with parentheses as above: $f(x) = \log_2(16) = 4$. This does not mean $\log_2$ times 16, but the output of the function $\log_2$ for the input of 16. Most of the time, however, people don't bother writing the parentheses unless they are necessary to specify the order or operations or to make the equation look prettier. I will generally omit them if the input to the function is simple, like *x* or *y*, but include them if the argument is more complex, just to be clear on what is being logged.

Non-integer exponents. In the examples given above, the exponents and therefore the logarithms were integers to make the relationships more intuitive. Quite often however, a logarithm will turn out to be a number with many places after the decimal point. For example, $log_{10}(25) = 1.39794$. If a logarithm is just an exponent, what is the meaning of these non-integer exponents?

Let's look at an example. According to my calculator, $3^{2.5} = 15.5885\ldots$. Not surprisingly, it is between $3^2 = 9$ and $3^3 = 27$. But what exactly does it mean? It helps to expand the equations:

$$3^2 = \left(3^1\right)\left(3^1\right) = (3)(3) = 9$$
$$3^3 = \left(3^1\right)\left(3^1\right)\left(3^1\right) = (3)(3)(3) = 27$$

The general pattern is $(a^x)(a^y) = a^{x+y}$ (remember Important Fact #1). Now just follow the same pattern for an exponent of 2.5, which is equal to 2 + 0.5:

$$3^{2.5} = 3^{(2+0.5)} = \left(3^2\right)\left(3^{0.5}\right) = 9(1.7321\ldots) = 15.5885\ldots$$

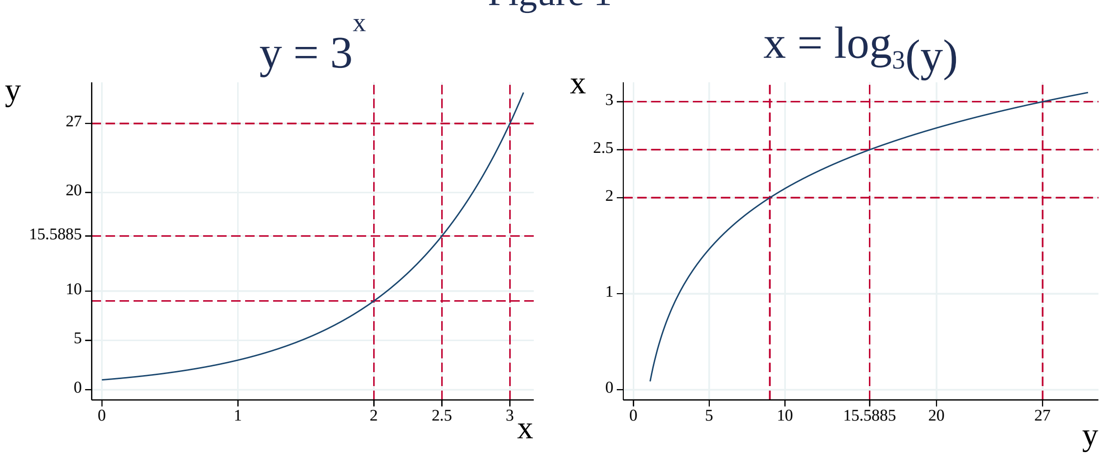
because the 0.5 power is the square root, and the square root of 3 is 1.7321….

Following Important Fact #3, $\left(a^x\right)^y = a^{xy}$, we could rewrite 3 raised to the 2.5 power in several equivalent ways:

$$3^{2.5} = 3^{\left(\frac{1}{2}\right)5} = \left(\sqrt{3}\right)^5 = \sqrt{3^5} = 15.5885...$$

In any case, the base 3 logarithm of 15.5885… is 2.5. Figure 1 shows 3 raised to the exponents of 2, 2.5, and 3 and the base 3 logarithms of the resulting values.

Figure 1



For *any* fractional power, we can compute the result by following a similar strategy. For example, what does $2^{2.87}$ mean? It's just $2^{2.87} = (2^2)(2^{0.87})$. That last term looks a little funky, but remember that 0.87 is just 87/100, so we can further express this as:

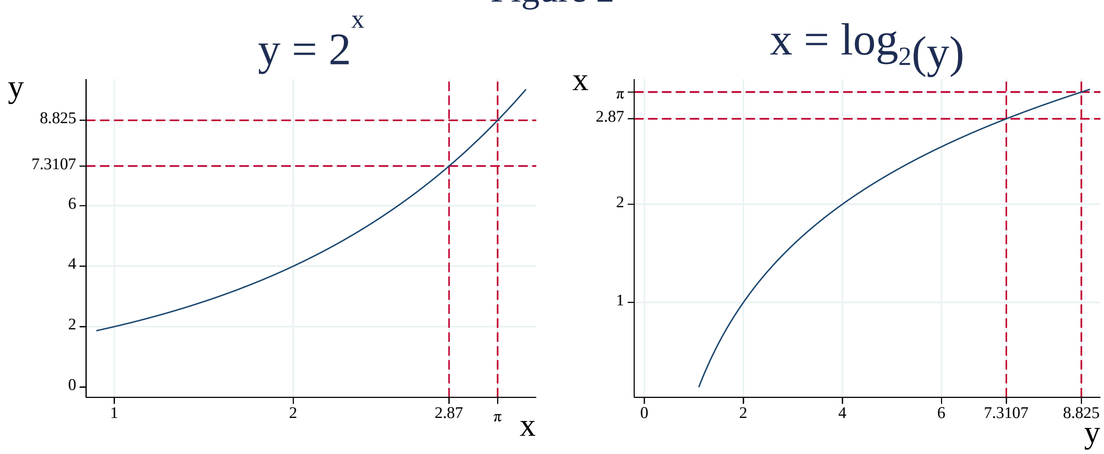
$$2^{2.87} = 2^{\left(2+\frac{87}{100}\right)} = \left(2^2\right)\left(\sqrt[100]{2^{87}}\right) = (4)(1.8277...) = 7.3107...$$

Pretty close to 2^3, but a little less. So, for any rational exponent, we can think of the result as a computation in involving integer powers and roots.

Irrational exponents are little more challenging. You have to think in terms of limits. To raise 2 to the π power, for example, we could say it's between $2^{3.1415}$ and $2^{3.1416}$, coming at it from above and below. As we add more digits of π, the upper and lower bound exponents approach ever closer to π, and we get closer and closer to the true value, 8.8249778…. ≈ 8.825.

The bottom line is that exponents, and therefore logarithms, define continuous functions can take on any value, not just integer values. Figure 2 shows 2 raised to the 2.87 and π and the base 2 logarithms of the resulting values.

Figure 2



All of this is to say that non-integer exponents are not weird or uninterpretable in any way. They mean exactly what they say they mean. And if fractional exponents make sense, then non-integer logarithms also make sense.

Bases. Bases also are not restricted to integers. Any positive real number can be the base of logarithms. Many different bases are used, especially in the sciences, but most of the time you are only going to see just two bases: 10 and e = 2.7182818284590452353602874713526624977572470936999595749669676277240766303535475945713821785251664274….

While e may seem like a strange number to use as a base for anything, it turns out that logarithms with this base have nice properties that make them easy to work with at higher levels of mathematics, e.g. calculus. Just think of e as a constant, like π = 3.14159…. Both are non-repeating infinite decimals, so it is convenient to use a symbol. If you need to perform a calculation, you first have to round it off.

Logarithms based on 10 are called *common logarithms*, while logarithms based on *e* are called *natural logarithms*. These two types of logarithms are usually written as follows:

$$\log_{10} x = \log x$$
$$\log_e x = \ln x$$

So if you see log( ) with no base specified, you should assume the base is 10 unless otherwise specified. If you see ln( ), the base is the number e.

Inverse Functions. Taking the base *b* logarithm and raising *b* to a power are inverse functions:

$$\log_b\left(b^a\right) = a \qquad b^{(\log_b a)} = a$$
$$\log_{10}\left(10^n\right) = n \qquad 10^{(\log_{10} n)} = n$$
$$\ln\left(e^x\right) = x \qquad e^{(\ln x)} = x$$

These identities just follow from the definition of logarithms. What power does *b* need to be raised to to get b^a? Well, just *a*. Take another look at the logarithm, base 5, of 5^2 above, the one that I said I would get back to.

Logarithms as Relative Changes. For *small changes*, the change in the logarithm of a variable is approximately the relative (or proportional) change in the underlying variable.[2] Multiply the change in the logarithm by 100, and it is approximately the percentage change. For example, a 5 percent increase in *x* results in almost a 0.05 increase in ln(*x*), regardless of the starting value of *x*. Example:

	x	ln(x)
	100	4.6052
	105	4.6540
Change	5 (5% increase)	0.0488
	12,345.00	9.4210
	12,962.25	9.4698
Change	617.25 (5% increase)	0.0488

Thus, if you have an interest in relative changes (change in proportional or percentage terms) rather than absolute changes, your analysis should focus on ln(*x*) as the dependent

[2] Calculus, if you know it, shows why:

$$\frac{d \ln x}{dx} = \frac{1}{x} \quad \rightarrow \quad d \ln x = \frac{dx}{x} \quad \rightarrow \quad \Delta \ln x \approx \frac{\Delta x}{x} \text{ for small changes}$$

variable. Again, this works best for small changes. It is best to use this interpretation for changes in the log of ±0.10 (±10 percent of *x*). The accuracy of this interpretation degrades the larger the change in the log.

3. Useful Properties of Logarithms.

The last section explained what logarithms are. This section explains why you should care. There are some properties of logarithms that turn out to be very useful for simplifying and manipulating equations. I will express these properties in terms of natural logarithms (ln = $\log_e$), but they work exactly the same for logarithms to any base. To express these properties for logarithms to another base *b*, just replace “ln” with “$\log_b$” and *e* with *b* in the formulas below. To express them for common logarithms, replace “ln” with “$\log_{10}$” (or just “log”) and *e* with 10.

Useful Property #1: First, *logarithms change multiplication into addition*:

$$\ln xy = \ln x + \ln y$$

Why? It’s just a consequence of the rule for multiplying powers described above. Let’s define two numbers, *m* and *n* that are the natural logarithms of *x* and *y*, respectively.

$$m = \ln x \quad \rightarrow \quad e^m = x$$
$$n = \ln y \quad \rightarrow \quad e^n = y$$

Now let’s look at the product of *x* and *y*:

$$xy = e^m e^n = e^{(m+n)}$$

Now we take the natural logarithm of both sides of the equation, preserving the equality:

$$\begin{aligned} \ln xy &= \ln e^{(m+n)} \\ &= m + n \\ &= \ln x + \ln y \end{aligned}$$

The changing of multiplication to addition is the basis of slide rules. A slide rule has two logarithmic scales; you slide one of the scales relative to the other so that you physically add up the two logarithms and read off the result. A lost art.

Useful Property #2: Second, *logarithms change division into subtraction*.

$$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$$

The proof is quite similar. Using the same definition of *m* and *n* as above:

$$\begin{aligned}
\frac{x}{y} &= \frac{e^m}{e^n} = e^{(m-n)} \\
\ln\left(\frac{x}{y}\right) &= \ln e^{(m-n)} \\
&= m - n \\
&= \ln x - \ln y
\end{aligned}$$

Useful Property #3: Third, *logarithms change exponents into multiplication*.

$$\ln\left(x^a\right) = a \ln x$$

To see this, we use the definition of *m* as the natural logarithm of *x* from above:

$$x^a = \left(e^m\right)^a = e^{ma}$$

This result follows from Important Fact #3 above. Now we log both sides:

$$\begin{aligned}
\ln\left(x^a\right) &= \ln\left(e^{ma}\right) \\
&= ma \\
&= a \ln x
\end{aligned}$$

Things you can't do. It is tempting sometimes to try to do things with logarithms that you just can't do. For example:

$$\ln(x+y) \neq \ln x + \ln y \qquad \frac{\ln x}{\ln y} \neq \ln\left(\frac{x}{y}\right)$$

$$\ln(x-y) \neq \ln x - \ln y$$

$$\ln\left(x^2\right) \neq (\ln x)^2 \qquad \ln x \ln y \neq \ln xy$$

Stick to the three properties described above and you won't slip into any of these errors.

Converting Bases. If you need to convert logarithms from one base to another, it is pretty easy. Here is the rule to convert a logarithm from base *a* to base *b*:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

For example, if we know that the natural log (ln = $\log_e$) of 100 is 4.60517, but we need the base 10 logarithm, we divide by the natural log of 10.

$$\log_{10} 100 = \frac{\ln 100}{\ln 10} = \frac{4.60517}{2.30258} = 2$$

We probably could have figured that out without the formula! But it illustrates the point. Here is where the base conversion formula comes from:

$$\log_a x = \log_a \left(b^{\log_b x}\right) = \left(\log_b x\right)\left(\log_a b\right)$$

$$\frac{\log_a x}{\log_a b} = \log_b x$$

Since the denominator is a constant, you can see that logs to one base are strictly proportional to logs in any other base.

4. Applications.

Linearizing a function. How can we use linear regression on non-linear functions?

$$Q = AL^{\beta_1} K^{\beta_2}$$

$$\ln Q = \ln A + \beta_1 \ln L + \beta_2 \ln K$$

Let ln*A* be the constant, add a disturbance term, and you can regress at will.

Problems with Interest Rates. How can we solve for the interest rate, *r*? For example, what is the rate of return if a $10,000 investment increases to 18,000 in 5 years? Starting with the basic compound interest formula, we can derive *r*.

$$X_n = X_0(1+r)^n$$
$$\ln X_n = \ln X_0 + n\ln(1+r)$$
$$\ln(1+r) = \frac{\ln X_n - \ln X_0}{n}$$
$$1+r = e^{\left(\frac{\ln X_n - \ln X_0}{n}\right)}$$
$$r = e^{\left(\frac{\ln X_n - \ln X_0}{n}\right)} - 1$$

I can never remember this formula, and so I have to re-derive it every time I need it. Good thing I know my logarithms. To answer the question posed above:

$$r = e^{\left(\frac{\ln X_n - \ln X_0}{n}\right)} - 1 = e^{\frac{\ln 18000 - \ln 10000}{6}} - 1 = e^{0.09796} - 1 = 1.103 - 1 = 10.3\%$$

Try solving for n and use that formula to see how many years it would take to double your money for a given interest rate.

5. Summary.

The table below summarizes the information above and shows the duality of the algebra of exponents and the properties of logarithms.

	Exponents	*Logarithms*
Definitions a, b, x, y all positive real numbers, $e = 2.718...$	$e^m = x, e^n = y$ $e^0 = 1$ $e^1 = e$ $e^x = e^x$	$m = \ln x, n = \ln y$ $\ln(1) = 0$ $\ln e = 1$ $\ln e^x = x, e^{\ln x} = x$
Multiplication	$xy = b^m b^n = b^{m+n}$	$\ln xy = \ln x + \ln y$
Division	$\frac{x}{y} = \frac{e^m}{e^n} = e^{m-n}$	$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$
Powers	$x^a = (e^m)^a = e^{ma}$	$\ln x^a = a \ln x$
Change of Base	$x = a^{\log_a x} = b^{\log_b x}$	$\log_a x = \frac{\log_b x}{\log_b a}$